Asymptotic formula for q-Derivative of q-Durrmeyer Operators

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Abstract

In the manuscript, Voronovskaja type asymptotic formula for function having q-derivative of q-Durrmeyer operators and q-Durrmeyer-Stancu operators are discussed.

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1. Introduction

The classical Bernstein-Durrmeyer operators D_n introduced by Durrmeyer [3] associate with each function f integrable on the interval $[0,1]$, the polynomial

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, \quad x \in [0,1], \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$.

These operators been studied by Derriennic [2] and many others. Last 30 years, the application of q-calculus in filed of approximation theory is active area of research. In 1987, the q-analogues of Bernstein operators was introduced by Lupas [10], Gupta and Hapeing [6] introduced q-generalization of the operators (1.1) as

$$D_{n,q}(f;x) = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{nk}(q;x)\int_0^1 f(t)p_{nk}(q;qt)d_qt, \tag{1.2}$$

where $p_{nk}(q;x) = \binom{n}{k}_q x^k(1-x)_q^{n-k}$.

The Rate of convergence of the operators (1.2) was discussed by Gupta *et al.* [5, 19], local approximation, global approximation and simultaneous approximation properties of these operators by Finta and Gupta [4], estimation of moments and King type approximation was elaborated by Gupta and Sharma [7]. In 2014, Mishra and Patel [12, 14] talk about Stancu generalization, Voronovskaja type asymptotic formula and various other approximation

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properties of the q-Durrmeyer-Stancu operators. We have the notation of q-calculus as given in [9, 17]. Here, in this manuscript we establish Voronovskaja type asymptotic formula for function having q-derivative.

2. Estimation of moments and Asymptotic formula

In the sequel, we shall need the following auxiliary results:

Theorem 1. *[7] If m-th $(m>0, m\in\mathbb{N})$ order moments of operator (1.2) is defined as*

$$D^q_{n,m}(x)=D_{n,q}(t^m,x)=[n+1]_q\sum_{k=0}^{n}q^{-k}p_{n,k}(q;x)\int_0^1 p_{n,k}(q;qt)t^m d_qt,\quad x\in[0,1],$$

then $D^q_{n,0}(x)=1$ and for $n>m+2$, we have following recurrence relation,

$$[n+m+2]_qD^q_{n,m+1}(x)=([m+1]_q+q^{m+1}x[n]_q)D^q_{n,m}(x)+x(1-x)q^{m+1}D^q(D^q_{n,m}(x)).$$

To establish asymptotic formula for function having q-derivative, it is necessary to compute moments of first to fourth degree. Using above Theorem one can have first, second, third and fourth order moments.

Lemma 1. *For all $x\in[0,1]$, $n=1,2,\dots$ and $0<q<1$, we have*

- $D_{n,q}(1,x)=1;$
- $D_{n,q}(t,x)=\dfrac{1+qx[n]_q}{[n+2]_q};$
- $D_{n,q}(t^2,x)=\dfrac{q^3x^2[n]_q[n-1]_q+(1+q)^2qx[n]_q+1+q}{[n+3]_q[n+2]_q};$
- $$D_{n,q}(t^3,x)=\frac{q^8x^3[n]_q[n-1]_q[n-2]_q+x^2q^3[n]_q[n-1]_q\left(1+q+2q^2+3q^3+2q^4\right)}{[n+4]_q[n+3]_q[n+2]_q}+\frac{xq[2]_q[n]_q\left(1+2q+3q^2+2q^3+q^4\right)+[3]_q[2]_q}{[n+4]_q[n+3]_q[n+2]_q};$$
- $$\begin{aligned}D_{n,q}(t^4,x)=&\frac{q^{15}x^4[n]_q[n-1]_q[n-2]_q[n-3]_q+q^8x^3[n]_q[n-1]_q[n-2]_q\left(1+2q+2q^2+3q^3+4q^4+3q^5+q^6\right)}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\\&+\frac{q^3x^2[n]_q[n-1]_q\left\{1+2q+4q^2+8q^3+12q^4+14q^5+13q^6+10q^7+6q^8+2q^9\right\}}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\\&+\frac{qx[2]_q[n]_q\left\{1+3q+6q^2+9q^3+10q^4+9q^5+6q^6+3q^7+q^8\right\}+[4]_q[3]_q[2]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}.\end{aligned}$$

Lemma 2. *For all $x\in[0,1]$, $n=1,2,\dots$ and $0<q<1$, we have*

- $D_{n,q}\left((t-x)_q,x\right)=\dfrac{1-\left(1+q^{n+1}\right)x}{[n+2]_q};$
- $D_{n,q}\left((t-x)^2_q,x\right)=\dfrac{q^2x^2(1+q^n)(q^{n+1}[2]_q-[n]_q)+x(1+q)(q^2[n]_q-1-q^{n+2})+1+q}{[n+3]_q[n+2]_q};$

- $D_{n,q}\left((t-x)_q^3,x\right)$
$$= q^2x^3\left\{\frac{q^6[n]_q[n-1]_q[n-2]_q - q[3]_q[n]_q[n-1]_q[n+4]_q + [n+4]_q[n+3]_q[2]_q[n]_q - q[n+4]_q[n+3]_q[n+2]_q}{[n+2]_q[n+3]_q[n+4]_q}\right\}$$
$$+ qx^2\left\{\frac{q^2[n]_q[n-1]_q\left(1+q+2q^2+3q^3+2q^4\right) - (1+q)^2[3]_q[n]_q[n+4]_q + [2]_q[n+4]_q[n+3]_q}{[n+2]_q[n+3]_q[n+4]_q}\right\}$$
$$+ x\left\{\frac{q[2]_q[n]_q\left(1+2q+3q^2+2q^3+q^4\right) - (1+q)[3]_q[n+4]_q}{[n+2]_q[n+3]_q[n+4]_q}\right\} + \frac{[3]_q[2]_q}{[n+2]_q[n+3]_q[n+4]_q};$$

- $D_{n,q}\left((t-x)_q^4,x\right)$
$$= x^4q^4\left\{\frac{q^{11}[n]_q[n-1]_q[n-2]_q[n-3]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q} - \frac{q^4[4]_q[n]_q[n-1]_q[n-2]_q}{[n+4]_q[n+3]_q[n+2]_q} + \frac{\left([5]_q+q^2\right)[n]_q[n-1]_q}{[n+3]_q[n+2]_q} - \frac{[4]_q[n]_q}{[n+2]_q} + q^2\right\}$$
$$+x^3q^2\left\{\frac{q^6[n]_q[n-1]_q[n-2]_q\left(1+2q+2q^2+3q^3+4q^4+3q^5+q^6\right)}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q} - \frac{q[4]_q[n]_q[n-1]_q\left(1+q+2q^2+3q^3+2q^4\right)}{[n+4]_q[n+3]_q[n+2]_q}\right.$$
$$\left.+\ \frac{(1+q)^2\left([5]_q+q^2\right)[n]_q}{[n+3]_q[n+2]_q} - q[4]_q\right\}$$
$$+ x^2\left\{\frac{q^2[n]_q[n-1]_q\left\{1+2q+4q^2+8q^3+12q^4+14q^5+13q^6+10q^7+6q^8+2q^9\right\}}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\right.$$
$$\left.-\frac{[4]_q[2]_q[n]_q\left(1+2q+3q^2+2q^3+q^4\right)}{[n+4]_q[n+3]_q[n+2]_q} + \frac{(1+q)\left([5]_q+q^2\right)}{[n+3]_q[n+2]_q}\right\} + \frac{[4]_q[3]_q[2]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}$$
$$+ x\left\{\frac{q[2]_q[n]_q\left\{1+3q+6q^2+9q^3+10q^4+9q^5+6q^6+3q^7+q^8\right\} + [4]_q[3]_q[2]_q[n+5]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\right\}.$$

Proof: To prove this Lemma, we use linear properties of q-Durrmeyer operators.

$$\begin{aligned} D_{n,q}((t-x)_q,x) &= D_{n,q}(t,x) - xD_{n,q}(1,x) = \frac{1+qx[n]_q}{[n+2]_q} - x = \frac{1+qx[n]_q - x[n+2]_q}{[n+2]_q} \\ &= \frac{1+x\left(q+q^2+\ldots+q^n-1-q-q^2-\ldots-q^n-q^{n+1}\right)}{[n+2]_q} \\ &= \frac{1-\left(1+q^{n+1}\right)x}{[n+2]_q}. \end{aligned}$$

Using identities $(t-x)_q^2 = t^2 - [2]_qxt + qx^2$, we get

$$\begin{aligned} D_{n,q}((t-x)_q^2,x) &= D_{n,q}(t^2,x) - [2]_qxD_{n,q}(t,x) + qx^2D_{n,q}(1,x) \\ &= \frac{q^3x^2[n]_q[n-1]_q + (1+q)^2qx[n]_q + 1 + q}{[n+3]_q[n+2]_q} - [2]_qx\left[\frac{1+qx[n]_q}{[n+2]_q}\right] + qx^2 \\ &= \frac{q^3x^2[n]_q[n-1]_q + (1+q)^2qx[n]_q + 1 + q - [2]_qx[n+3]_q - qx^2[2]_q[n+3]_q[n]_q + qx^2[n+3]_q[n+2]_q}{[n+3]_q[n+2]_q} \\ &= \frac{qx^2\left\{q^2[n]_q[n-1]_q - [2]_q[n+3]_q[n]_q + [n+3]_q[n+2]_q\right\} + x\left\{(1+q)^2q[n]_q - [2]_q[n+3]_q\right\} + 1 + q}{[n+3]_q[n+2]_q} \\ &= \frac{q^2x^2(1+q^n)(q^{n+1}[2]_q - [n]_q) + x(1+q)(q^2[n]_q - 1 - q^{n+2}) + 1 + q}{[n+3]_q[n+2]_q}. \end{aligned}$$

Notice that $(t-x)_q^3 = t^3 - [3]_q x t^2 + q[2]_q x^2 t - q^3 x^3$,

$$\begin{aligned}
D_{n,q}\left((t-x)_q^3, x\right) &= D_{n,q}\left(t^3, x\right) - [3]_q x D_{n,q}\left(t^2, x\right) + q[2]_q x^2 D_{n,q}(t, x) - q^3 x^3 \\
&= \frac{q^8 x^3 [n]_q [n-1]_q [n-2]_q + x^2 q^3 [n]_q [n-1]_q \left(1+q+2q^2+3q^3+2q^4\right)}{[n+4]_q [n+3]_q [n+2]_q} \\
&\quad + \frac{x q [2]_q [n]_q \left(1+2q+3q^2+2q^3+q^4\right) + [3]_q [2]_q}{[n+4]_q [n+3]_q [n+2]_q} \\
&\quad - [3]_q x \left\{ \frac{q^3 x^2 [n]_q [n-1]_q + (1+q)^2 q x [n]_q + 1 + q}{[n+3]_q [n+2]_q} \right\} + q[2]_q x^2 \left\{ \frac{1 + q x [n]_q}{[n+2]_q} \right\} - q^3 x^3 \\
&\quad + q x^2 \left\{ \frac{q^2 [n]_q [n-1]_q \left(1+q+2q^2+3q^3+2q^4\right) - (1+q)^2 [3]_q [n]_q [n+4]_q + [2]_q [n+4]_q [n+3]_q}{[n+2]_q [n+3]_q [n+4]_q} \right\} \\
&\quad + x \left\{ \frac{q[2]_q [n]_q \left(1+2q+3q^2+2q^3+q^4\right) - (1+q)[3]_q [n+4]_q}{[n+2]_q [n+3]_q [n+4]_q} \right\} + \frac{[3]_q [2]_q}{[n+2]_q [n+3]_q [n+4]_q}.
\end{aligned}$$

Finally, using identities $(t-x)_q^4 = t^4 - [4]_q x t^3 + q\left([5]_q + q^2\right) x^2 t^2 - q^3 x^3 [4]_q t + q^6 x^4$, we get

$$\begin{aligned}
& D_{n,q}\left((t-x)_q^4, x\right) \\
= \; & D_{n,q}\left(t^4, x\right) - [4]_q x D_{n,q}\left(t^3, x\right) + q\left([5]_q + q^2\right) x^2 D_{n,q}\left(t^2, x\right) - q^3 x^3 [4]_q D_{n,q}(t, x) + q^6 x^4 \\
= \; & \frac{q^{15} x^4 [n]_q [n-1]_q [n-2]_q [n-3]_q + q^8 x^3 [n]_q [n-1]_q [n-2]_q \left(1+2q+2q^2+3q^3+4q^4+3q^5+q^6\right)}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} \\
& + \frac{q^3 x^2 [n]_q [n-1]_q \left\{1+2q+4q^2+8q^3+12q^4+14q^5+13q^6+10q^7+6q^8+2q^9\right\}}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} \\
& + \frac{q x [2]_q [n]_q \left\{1+3q+6q^2+9q^3+10q^4+9q^5+6q^6+3q^7+q^8\right\} + [4]_q [3]_q [2]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} \\
& - [4]_q x \left\{ \frac{q^8 x^3 [n]_q [n-1]_q [n-2]_q + x^2 q^3 [n]_q [n-1]_q \left(1+q+2q^2+3q^3+2q^4\right)}{[n+4]_q [n+3]_q [n+2]_q} \right. \\
& \left. + \frac{x q [2]_q [n]_q \left(1+2q+3q^2+2q^3+q^4\right) + [3]_q [2]_q}{[n+4]_q [n+3]_q [n+2]_q} \right\} \\
& + q\left([5]_q + q^2\right) x^2 \left\{ \frac{q^3 x^2 [n]_q [n-1]_q + (1+q)^2 q x [n]_q + 1 + q}{[n+3]_q [n+2]_q} \right\} - q^3 x^3 [4]_q \left\{ \frac{1 + q x [n]_q}{[n+2]_q} \right\} + q^6 x^4 \\
= \; & \frac{q^{15} x^4 [n]_q [n-1]_q [n-2]_q [n-3]_q + q^8 x^3 [n]_q [n-1]_q [n-2]_q \left(1+2q+2q^2+3q^3+4q^4+3q^5+q^6\right)}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} \\
& + \frac{q^3 x^2 [n]_q [n-1]_q \left\{1+2q+4q^2+8q^3+12q^4+14q^5+13q^6+10q^7+6q^8+2q^9\right\}}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} \\
& + \frac{q x [2]_q [n]_q \left\{1+3q+6q^2+9q^3+10q^4+9q^5+6q^6+3q^7+q^8\right\} + [4]_q [3]_q [2]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} \\
& - \frac{q^8 [4]_q x^4 [n+5]_q [n]_q [n-1]_q [n-2]_q + q^3 [4]_q x^3 [n+5]_q [n]_q [n-1]_q \left(1+q+2q^2+3q^3+2q^4\right)}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} \\
& - \frac{q [4]_q [2]_q x^2 [n+5]_q [n]_q \left(1+2q+3q^2+2q^3+q^4\right) + x [4]_q [3]_q [2]_q [n+5]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{q^4\left([5]_q+q^2\right)x^4[n+5]_q[n+4]_q[n]_q[n-1]_q+q^2(1+q)^2\left([5]_q+q^2\right)x^3[n+5]_q[n+4]_q[n]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\\
&+\frac{(1+q)q\left([5]_q+q^2\right)x^2[n+5]_q[n+4]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\\
&-\frac{q^3x^3[4]_q[n+5]_q[n+4]_q[n+3]_q+q^4x^4[4]_q[n+5]_q[n+4]_q[n+3]_q[n]_q-q^6x^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\\
= \quad & x^4q^4\left\{\frac{q^{11}[n]_q[n-1]_q[n-2]_q[n-3]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}-\frac{q^4[4]_q[n]_q[n-1]_q[n-2]_q}{[n+4]_q[n+3]_q[n+2]_q}+\frac{\left([5]_q+q^2\right)[n]_q[n-1]_q}{[n+3]_q[n+2]_q}-\frac{[4]_q[n]_q}{[n+2]_q}+q^2\right\}\\
&+x^3q^2\left\{\frac{q^6[n]_q[n-1]_q[n-2]_q\left(1+2q+2q^2+3q^3+4q^4+3q^5+q^6\right)}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}-\frac{q[4]_q[n]_q[n-1]_q\left(1+q+2q^2+3q^3+2q^4\right)}{[n+4]_q[n+3]_q[n+2]_q}\right.\\
&\left.+\frac{(1+q)^2\left([5]_q+q^2\right)[n]_q}{[n+3]_q[n+2]_q}-q[4]_q\right\}\\
&+x^2\left\{\frac{q^2[n]_q[n-1]_q\left\{1+2q+4q^2+8q^3+12q^4+14q^5+13q^6+10q^7+6q^8+2q^9\right\}}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\right.\\
&\left.-\frac{[4]_q[2]_q[n]_q\left(1+2q+3q^2+2q^3+q^4\right)}{[n+4]_q[n+3]_q[n+2]_q}+\frac{(1+q)\left([5]_q+q^2\right)}{[n+3]_q[n+2]_q}\right\}+\frac{[4]_q[3]_q[2]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\\
&+x\left\{\frac{q[2]_q[n]_q\left\{1+3q+6q^2+9q^3+10q^4+9q^5+6q^6+3q^7+q^8\right\}+[4]_q[3]_q[2]_q[n+5]_q}{[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\right\}.
\end{aligned}
$$

Theorem 2. *Let f bounded and integrable on the interval $[0,1]$ and (q_n) denote a sequence such that $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$. Then we have for a point $x\in(0,1)$*

$$\lim_{n\to\infty}[n]_{q_n}[D_{n,q_n}(f;x)-f(x)]=(1-2x)\lim_{n\to\infty}D_{q_n}f(x)+x(1-x)\lim_{n\to\infty}D^2_{q_n}f(x).$$

Proof: By q-Taylor formula [1] for f, we have

$$f(t)=f(x)+D_qf(x)(t-x)+\frac{1}{[2]_q}D_q^2f(x)(t-x)_q^2+\theta_q(x;t)(t-x)_q^2,$$

for $0<q<1$, where

$$\theta_q(x;t)=\begin{cases}\dfrac{f(t)-f(x)-D_qf(x)(t-x)-\frac{1}{[2]_q}D_q^2f(x)(t-x)_q^2}{(t-x)_q^2} & \text{if } x\neq t\\ 0, & \text{if } x=t.\end{cases} \tag{2.1}$$

We know that for n large enough

$$\lim_{t\to x}\theta_q(x;t)=0. \tag{2.2}$$

That is for any $\epsilon>0$, there exists a $\delta>0$ such that

$$|\theta_q(x;t)|\le\epsilon. \tag{2.3}$$

for $|t-x|<\delta$ and n sufficiently large. Using (2.1), we can write

$$D_{n,q_n}(f;x)-f(x)=D_{q_n}f(x)D_{n,q_n}((t-x)_q;x)+\frac{D^2_{q_n}f(x)}{[2]_{q_n}}D_{n,q_n}((t-x)^2_q;x)+E^{q_n}_n(x),$$

where

$$E^q_n(x)=[n+1]_q\sum_{k=0}^n q^{-k}p_{nk}(q;x)\int_0^1\theta_q(x;t)p_{nk}(q;qt)\,(t-x)^2_q\,d_qt.$$

By Lemma 2, we have

$$\lim_{n\to\infty}[n]_{q_n}D_{n,q_n}((t-x)_q;x)=(1-2x)\text{ and }\lim_{n\to\infty}[n]_{q_n}D_{n,q_n}((t-x)^2_q;x)=2x(1-x).$$

In order to complete the proof of the theorem, it is sufficient to show that $\lim_{n\to\infty}[n]_{q_n}E^{q_n}_n(x)=0$. We proceed as follows:

Let

$$P^{q_n}_{n,1}(x)=[n]_{q_n}[n+1]_{q_n}\sum_{k=0}^n {q_n}^{-k}p_{nk}(q_n;x)\int_0^1\theta_{q_n}(x;t)p_{nk}(q_n;q_nt)\,(t-x)^2_{q_n}\,\chi_x(t)d_{q_n}t$$

and

$$P^{q_n}_{n,2}(x)=[n]_{q_n}[n+1]_{q_n}\sum_{k=0}^n {q_n}^{-k}p_{nk}(q_n;x)\int_0^1\theta_{q_n}(x;t)p_{nk}(q_n;q_nt)\,(t-x)^2_{q_n}\,(1-\chi_x(t))\,d_{q_n}t,$$

so that

$$[n]_{q_n}E^{q_n}_n(x)=P^{q_n}_{n,1}(x)+P^{q_n}_{n,2}(x),$$

where $\chi_x(t)$ is the characteristic function of the interval $\{t:|t-x|<\delta\}$.

It follows from (2.1)

$$P^{q_n}_{n,1}(x)=2\epsilon x(1-x)\text{ as }n\to\infty.$$

If $|t-x|\geq\delta$, then $|\theta_{q_n}(x;t)|\leq\frac{M}{\delta^2}(t-x)^2$, where $M>0$ is a constant. Since

$$\begin{aligned}(t-x)^2&=\left(t-q^2x+q^2x-x\right)\left(t-q^3x+q^3x-x\right)\\&=\left(t-q^2x\right)\left(t-q^3x\right)+x(q^3-1)\left(t-q^2x\right)+x(q^2-1)\left(t-q^2x\right)+x^2(q^2-1)(q^2-q^3)+x^2(q^2-1)(q^3-1),\end{aligned}$$

we have

$$\begin{aligned}|P^{q_n}_{n,2}(x)|&\leq\frac{M}{\delta^2}\left\{[n]_{q_n}D_{n,q_n}((t-x)^4_{q_n};x)+x(2-q_n^2-q_n^3)[n]_{q_n}D_{n,q_n}((t-x)^3_{q_n};x)\right.\\&\quad\left.+x^2(q_n^2-1)^2[n]_{q_n}D^{\alpha,\beta}_{n,q_n}((t-x)^2_{q_n};x)\right\}.\end{aligned}$$

Using Lemma 2, we have

$$D_{n,q_n}((t-x)^4_{q_n};x)\leq\frac{C_m}{[n]^3_{q_n}},\quad D_{n,q_n}((t-x)^3_{q_n};x)\leq\frac{C_m}{[n]^2_{q_n}}\quad\text{and}\quad D_{n,q_n}((t-x)^2_{q_n};x)\leq\frac{C_m}{[n]_{q_n}},$$

we have the desired result.

Corollary 1. *Let f bounded and integrable on the interval $[0,1]$ and (q_n) denote a sequence such that $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$. Suppose that the first and second derivative $f'(x)$ and $f''(x)$ exist at a point $x\in(0,1)$. Then we have for a point $x\in(0,1)$*

$$\lim_{n\to\infty}[n]_{q_n}[D_{n,q_n}(f;x)-f(x)]=(1-2x)f'(x)+x(1-x)f''(x).$$

3. Asymptotic formula for Durrmeyer-Stancu Operators

In year 1968, Stancu [16] generalized Bernstein operators and discussed it approximation properties. After that numbers of researchers gives Stancu type generalization of several operators on finite and infinite intervals, we refer to the papers [13, 11, 15, 8, 18]. As mention in the introduction Stancu generalization of q-Durrmeyer operators (1.2) was discussed by Mishra and Patel [12], which is defined as follows: for $0\le\alpha\le\beta$,

$$D_{n,q}^{\alpha,\beta}=[n+1]_q\sum_{k=0}^{n}q^{-k}p_{nk}(q;x)\int_0^1 f\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)p_{nk}(q;qt)d_qt, \tag{3.1}$$

where $p_{nk}(q;x)$ as same as defined in (1.2).

Lemma 3. *We have* $D_{n,q}^{\alpha,\beta}(1;x)=1,\qquad D_{n,q}^{\alpha,\beta}(t;x)=\dfrac{[n]_q+\alpha[n+2]_q+qx[n]_q^2}{[n+2]_q\left([n]_q+\beta\right)},$

$$\begin{aligned}D_{n,q}^{\alpha,\beta}(t^2;x)=&\frac{q^3[n]_q^3\left([n]_q-1\right)x^2+\left(\left(q(1+q)^2+2\alpha q^4\right)[n]_q^3+2\alpha q[3]_q[n]_q^2\right)x}{([n]_q+\beta)^2[n+2]_q[n+3]_q}\\&+\frac{\alpha^2}{([n]_q+\beta)^2}+\frac{(1+q+2\alpha q^3)[n]_q^2+2\alpha[3]_q[n]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q}.\end{aligned}$$

Lemma 4. *We have*

$$D_{n,q}^{\alpha,\beta}(t-x,x)=\left(\frac{q[n]_q^2}{[n+2]_q([n]_q+\beta)}-1\right)x+\frac{[n]_q+\alpha[n+2]_q}{[n+2]_q([n]_q+\beta)},$$

$$\begin{aligned}D_{n,q}^{\alpha,\beta}((t-x)^2,x)=&\frac{q^4[n]_q^4-q^3[n]_q^3-2q[n]_q^2[n+3]_q([n]_q+\beta)+[n+2]_q[n+3]_q([n]_q+\beta)^2}{([n]_q+\beta)^2[n+2]_q[n+3]_q}x^2\\&+\frac{q(1+q)^2[n]_q^3+2q\alpha[n]_q^2[n+3]_q-\left(2[n]_q+2\alpha[n+2]_q\right)[n+3]_q([n]_q+\beta)}{([n]_q+\beta)^2[n+2]_q[n+3]_q}x\\&+\frac{(1+q)[n]_q^2+2\alpha[n]_q[n+3]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q}.\end{aligned}$$

Remark 1. *For all $m\in\mathbf{N}\cup\{0\}, 0\le\alpha\le\beta$; we have the following recursive relation for the images of the monomials t^m under $D_{n,q}^{\alpha,\beta}(t^m;x)$ in terms of $D_{n,q}(t^j;x); j=0,1,2,\ldots,m$, as*

$$D_{n,q}^{\alpha,\beta}(t^m;x)=\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\alpha^{m-j}}{([n]_q+\beta)^m}D_{n,q}(t^j,x).$$

Theorem 3. *Let f bounded and integrable on the interval $[0,1]$ and (q_n) denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Then we have for a point $x \in (0,1)$*

$$\lim_{n\to\infty} [n]_{q_n} [D_{n,q_n}^{\alpha,\beta}(f;x) - f(x)] = (1 + \alpha - (2+\beta)x) \lim_{n\to\infty} D_{q_n} f(x) + x(1-x) \lim_{n\to\infty} D_{q_n}^2 f(x).$$

The proof of the above lemma follows along the lines of Theorem 2, using Lemma 4 and remark 1; thus, we omit the details.

Corollary 2. *[12] Let f bounded and integrable on the interval $[0,1]$ and (q_n) denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Suppose that the first and second derivative $f'(x)$ and $f''(x)$ exist at a point $x \in (0,1)$. Then we have for a point $x \in (0,1)$*

$$\lim_{n\to\infty} [n]_{q_n} [D_{n,q_n}^{\alpha,\beta}(f;x) - f(x)] = (1 + \alpha - (2+\beta)x) f'(x) + x(1-x) f''(x).$$

Remark 2. *Theorem 2 and Theorem 3, gives asymptotic formula for q-Durrmeyer operators and q-Durrmeyer-Stancu operators respectively. If f has first and second derivative, then $\lim_{n\to\infty} D_{q_n} f(x) = f'(x)$ and $\lim_{n\to\infty} D_{q_n}^2 f(x) = f''(x)$. We archived results of Mishra and Patel [12, Theorem 5], which is mention in corollary 2. So presented results are more general results then exists ones.*

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